Control of a Nonaffine Double-Pendulum System via Dynamic Inversion and Time-Scale Separation

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Abstract. A new method is presented for control of nonaffine double pendulum system via dynamic inversion and time-scale separation. The control signal is defined as a solution of “fast” dynamics, and the coupled system is shown to be stable using the framework of Tikhonov’s theorem from singular perturbations theory. Simulations illustrate the theoretical results.

I. INTRODUCTION

Stabilization of the inverted pendulum is considered to be one of the benchmark problems in nonlinear control [1]–[5]. This model is mainly used for demonstration of various new control methodologies, like genetic algorithms in [1], partial feedback linearization and integrator backstepping in [5], energy based approach in [6], to name a few. The problem of stabilization becomes more challenging, when the pendulum is comprised of more than one rod [7]–[13]. Practical significance of these studies is apparent in a variety of realistic applications, like control of manipulators or crane control [14]–[16]. In this particular paper, we are interested in using the double-pendulum as an academic example to demonstrate a novel control methodology for nonaffine nonlinear systems. Control of nonaffine-in-input pendulum has been considered in [4]. The significance of nonaffine-in-control methods is apparent when considering applications, in which the system’s control effectiveness depends nonlinearly upon the control input [17].

In this paper, we use the control methodology from [18] to solve the tracking problem for a nonaffine double inverted pendulum. The method relies on time-scale separation between the system dynamics and the controller dynamics. The control signal is sought as solution of fast dynamics, which is shown to satisfy the sufficient conditions of Tikhonov’s theorem from singular perturbations theory.

The paper is organized as follows. In Section II, we describe the dynamics and give the problem formulation. Section III details the control design, which is further analyzed in Section IV. Simulation results are given in Section V.

II. PROBLEM STATEMENT

Consider the two degree-of-freedom double pendulum in Figure 1. The two rods may rotate in the vertical plane.

Torque control is applied at the two connecting joints. The rods are treated as rigid bodies, and all frictional forces are ignored. The following equations of motion can be derived via Lagrange’s equations [19]:

\[
M_1(t) - M_2(t) = \frac{1}{2} l_2^2 (m_1 + 3m_2) \ddot{\theta}(t) + \frac{1}{2} gl_1 (m_1 + 2m_2) \sin(\theta)(t) + \frac{1}{2} l_1 l_2 m_2 \cos(\phi(t) - \theta(t)) \ddot{\phi}(t) - \frac{1}{2} l_1 l_2 m_2 \dot{\phi}^2(t) \sin(\phi(t) - \theta(t))
\]

\[
M_2(t) = \frac{1}{2} l_1 l_2 m_2 \cos(\phi(t) - \theta(t)) \ddot{\phi}(t) + \frac{1}{2} m_2 l_2^2 \dot{\phi}(t) + \frac{1}{2} gl_2 m_2 \sin(\phi)(t) - \frac{1}{2} l_1 l_2 m_2 \dot{\phi}^2(t) \sin(\phi(t) - \theta(t)),
\]

where \(\theta(t)\) and \(\phi(t)\) are the generalized coordinates, and \(M_1(t)\) and \(M_2(t)\) are the torques acting on the connecting joints of rod 1 and rod 2. In general, torque is produced via an actuator that can nonlinearly depend upon the system states and the actual control input, so that \(M_i(t) = \bar{M}_i(\theta(t), \phi(t), u_i(t))\). With the following notations

\[
f_{11}(\theta, \phi) = \frac{12}{l_1^2 [4m_1 + 12m_2 + 9m_2 \cos^2(\phi - \theta)]},
\]

\[
f_{12}(\theta, \phi) = \frac{12l_2 + 18l_1 \cos(\phi - \theta)}{l_1^2 [9m_2 \cos^2(\phi - \theta) - 4m_1 - 12m_2]},
\]

\[
f_{21}(\theta, \phi) = \frac{18 \cos(\phi - \theta)}{l_1 l_2 [9m_2 \cos^2(\phi - \theta) - 4m_1 - 12m_2]},
\]
f_{22}(\theta, \phi) = \frac{12 \text{m}_1 + 36 \text{m}_2}{l_2^2 \text{m}_2 [4 \text{m}_1 + 12 \text{m}_2 - 9 \text{m}_2 \cos^2 (\phi - \theta)]} + \frac{18 \text{g} \text{m}_2 \cos (\phi - \theta)}{l_1 l_2^2 [4 \text{m}_1 + 12 \text{m}_2 - 9 \text{m}_2 \cos^2 (\phi - \theta)]},

f_{13}(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{9 \text{g} \text{m}_2 \cos (2 \phi - \theta)}{l_1 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} - \frac{9 \text{l}_1 \text{m}_2 \dot{\theta}^2 \cos (2 \phi - \theta)}{l_1 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} + \frac{12 \text{l}_2 \text{m}_2 \dot{\phi}^2 \sin (\phi - \theta)}{l_1 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} - \frac{15 \text{g} \text{m}_2 + 12 \text{g} \text{m}_1 \sin (\phi)}{l_1 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]}.

f_{23}(\theta, \dot{\theta}, \phi, \dot{\phi}) = \frac{12 \text{l}_1 \dot{\theta}^2 \sin (\phi - \theta) (\text{m}_1 + 3 \text{m}_2)}{l_2 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} - \frac{9 \text{l}_1 \text{m}_2 \dot{\theta}^2 \sin (2 \phi - \theta)}{l_2 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} - \frac{9 \text{g} \sin (\phi - \theta) (\text{m}_1 + 2 \text{m}_2)}{l_2 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)]} + \frac{3 \text{g} \sin (\phi) (\text{m}_1 + 6 \text{m}_2)}{l_2 [15 \text{m}_2 + 8 \text{m}_1 - 9 \text{m}_2 \cos (2 \phi - 2 \theta)].

The equations in (1) can be solved with respect to \( \dot{\theta}(t) \) and \( \dot{\phi}(t) \):

\[
\begin{align*}
\dot{\theta}(t) &= f_{11}(\theta, \phi) M_1(\theta(t), \phi(t), u_1(t)) + f_{12}(\theta, \phi) M_2(\theta(t), \phi(t), u_2(t)) + f_{13}(\theta, \phi, \dot{\phi}) \\
\dot{\phi}(t) &= f_{21}(\theta, \phi) M_1(\theta(t), \phi(t), u_1(t)) + f_{22}(\theta, \phi) M_2(\theta(t), \phi(t), u_2(t)) + f_{23}(\theta, \phi, \dot{\phi}),
\end{align*}
\]

(2)

where \( \theta_0, \dot{\theta}_0, \phi_0, \dot{\phi}_0 \) are initial conditions. Letting \( x_1(t) = \theta(t), \ x_2(t) = \dot{\theta}(t), \ x_3(t) = \phi(t), \ x_4(t) = \dot{\phi}(t), \) the double pendulum system in (1) can be written in state-space form:

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\dot{x}_3(t) \\
\dot{x}_4(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix}
+ \begin{bmatrix}
0 \\
1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
f_1(x(t), u(t)) \\
f_2(x(t), u(t))
\end{bmatrix},
\]

\[
y(t) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t)
\end{bmatrix}, \quad x(0) = x_0,
\]

where \( x(t) = [x_1(t) \ x_2(t) \ x_3(t) \ x_4(t)]^T \) is the vector of measurable states, \( y(t) = [y_1(t) \ y_2(t)]^T \) is the vector of regulated outputs, \( u(t) = [u_1(t) \ u_2(t)]^T \) is the vector of control inputs, and

\[
f_1(x, u) = f_{11}(x) M_1(x, u_1) + f_{12}(x) M_2(x, u_2) + f_{13}(x)
\]

(3)

\[
f_2(x, u) = f_{21}(x) M_1(x, u_1) + f_{22}(x) M_2(x, u_2) + f_{23}(x).
\]

It is straightforward to verify that \( f_{11}, f_{12}, f_{13}, f_{21}, f_{22}, f_{23} \) are well-defined for all \( x \in \mathbb{R}^4 \). We notice that the system has full relative degree with respect to the regulated outputs so that there are no internal dynamics present in the system. To ensure controllability, we assume that the nonlinear functions of control input satisfy:

\[
\left| \frac{\partial M_i(x, u_i)}{\partial u_i} \right| \geq \beta_i > 0
\]

(4)

for some \( \beta_i > 0 \) and \( i = 1, 2 \). The control objective is to design control input \( u(t) \) so that the output \( y(t) \) tracks a desired bounded reference trajectory \( r_d(t) \) asymptotically.

### III. CONTROL DESIGN FOR THE PENDULUM SYSTEM

#### A. Reference Model

Consider a reference system given by:

\[
\begin{align*}
\dot{x}_r(t) &= A_r x_r(t) + B_r r(t), \quad t \geq 0, \quad x_r(0) = x_{r,0} \\
y_r(t) &= C_r x_r(t),
\end{align*}
\]

where the matrices \( A_r, B_r, \) and \( C_r \) are defined as

\[
A_r = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-\alpha_{r3} & -\alpha_{r4} & -\alpha_{r3} & -\alpha_{r4}
\end{bmatrix},
\]

\[
B_r = \begin{bmatrix}
b_{r3} & 0 \\
0 & b_{r4}
\end{bmatrix}, \quad C_r = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
x_r(t) = \begin{bmatrix}
x_1^r(t) \\
x_2^r(t) \\
x_3^r(t) \\
x_4^r(t)
\end{bmatrix}^T
\]

denotes the state vector of the reference system, \( \alpha_{r1}, \alpha_{r2}, \alpha_{r3}, \alpha_{r4} \) are such that \( A_r \) is Hurwitz. Let \( r_d(t) = [r_{d1}(t) \ r_{d2}(t)]^T \) be the vector of desired smooth bounded reference trajectories. Due to the time-varying nature of \( r_d(t) \) and the definition of the reference model, if one sets \( r(t) = r_d(t) \), then \( y_r(t) \) will track \( r_d(t) \) with bounded errors. To force \( y_r(t) \) track the desired signal \( r_d(t) \) asymptotically and guarantee zero steady state tracking error, we introduce the following reference input:

\[
r(t) = \begin{bmatrix}
\alpha_{r1} & 0 & 0 & 0 \\
\beta_{r1} & \alpha_{r2} & 0 & 0 \\
0 & \alpha_{r3} & \alpha_{r4}
\end{bmatrix} r_d(t) + \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} \dot{r}_d(t) + \begin{bmatrix}
\frac{1}{\beta_{r1}} \\
0
\end{bmatrix} \dddot{r}_d(t).
\]

(6)
It is straightforward to verify that \( \lim_{t \to \infty} (y_r(t) - r_d(t)) = 0 \), where \( y_r(t) \) has been defined in (5).

### B. Control Law

Let \( e(t) = x(t) - x_r(t) \) be the tracking error. Then the open-loop tracking error dynamics are given by:

\[
e(t) = F(e(t) + x_r(t), u(t)) - A_r x_r(t) - B_r r(t),
\]

with \( e(0) = e_0 \) and

\[
F(x, u) = \begin{bmatrix} f_1(x, u) & f_2(x, u) \end{bmatrix}^\top,
\]

where \( f_i(x, u) \) have been defined in (3). Dynamic inversion based control is to be determined from the solution of the following system of equations

\[
\begin{bmatrix}
  f_1(x, u) \\
  f_2(x, u)
\end{bmatrix} = \begin{bmatrix}
  -a_2 x_1 + a_2 x_2 + b_2 r_1 \\
  -a_4 x_3 - a_4 x_4 + b_4 r_2
\end{bmatrix},
\]

yielding the following asymptotically stable error dynamics:

\[
\dot{e}(t) = A_r e(t), \quad e(0) = e_0.
\]

However, due to the nonlinear expressions in (3), an explicit expression for \( u \) from (8) cannot be obtained, in general. Following [18], introduce the following fast dynamics to approximate its solution:

\[
e\dot{u}(t) = -\Lambda(t, e(t), u(t))Q^{-1}(t, e(t))f(t, e(t), u(t)) \quad (9)
\]

where

\[
f(t, e, u) = \begin{bmatrix}
  f_1(e + x_r(t), u) + a_{21} (x_1^2 + e_1) + a_{22} (x_2^2 + e_2) - b_21 r_1(t) \\
  f_2(e + x_r(t), u) + a_{44} (x_4^2 + e_4) - b_42 r_2(t)
\end{bmatrix},
\]

\[
Q(t, e) = \begin{bmatrix}
  f_{11}(e + x_r(t)) & f_{12}(e + x_r(t)) \\
  f_{21}(e + x_r(t)) & f_{22}(e + x_r(t))
\end{bmatrix},
\]

and

\[
\Lambda(t, e, u) = \begin{bmatrix}
  \frac{\partial M_1(e+x_r(t), u)}{\partial u_1} & 0 \\
  0 & \frac{\partial M_2(e+x_r(t), u)}{\partial u_2}
\end{bmatrix}.
\]

We also notice that the matrix \( Q(x) \) can be row reduced using the explicit forms of \( f_{11}(x), f_{12}(x), f_{21}(x) \) and \( f_{22}(x) \) and rewritten as

\[
\begin{bmatrix}
  1 & q_1(x) \\
  0 & q_2(x)
\end{bmatrix},
\]

in which

\[
q_2(x) = 4l_1 m_1 + 12l_1 m_2 - 9l_1 m_2 \cos^2(x_3 - x_1) \neq 0
\]

for any \( x \in \mathbb{R}^4 \), which implies that \( Q(x) \) is full rank, so that its inverse, used in (9), is well-defined. We argue that the solution of the singularly perturbed system (7), (9) will solve the tracking problem. Towards that end, we recall Tikhonov’s theorem from singular perturbations.

### IV. Convergence Analysis of the Nonaffine Control Law

#### A. Tikhonov’s Theorem

Consider the problem of solving the following singularly perturbed system

\[
\begin{align*}
\Sigma_0 : & \quad \dot{x}(t) = f(t, x(t), u), \quad x(0) = \zeta(0) \\
\Sigma_0 : & \quad \dot{e}(t) = g(t, x(t), u), \quad e(0) = \eta(0)
\end{align*}
\]

where \( \zeta : \epsilon \mapsto \zeta(\epsilon) \) and \( \eta : \epsilon \mapsto \eta(\epsilon) \) are smooth, \( f \) and \( g \) are continuously differentiable in their arguments for \( (t, x, u) \in [0, \infty) \times D_x \times D_u \times [0, \epsilon] \), where \( D_x \subset \mathbb{R}^n \) and \( D_u \subset \mathbb{R}^m \) are domains, and \( \epsilon_0 > 0 \). Let \( \Sigma_0 \) be in standard form, that is,

\[
x = g(t, x, u, 0)
\]

(15)

has \( k \geq 1 \) isolated real roots \( u = h_i(t, x), \quad i \in \{1, \ldots, k\} \) for any \( (t, x) \in [0, \infty) \times D_x \). Choose one particular \( i \) and drop the subscript henceforth. Let \( \nu(t, x) = u - h(t, x) \). Then the system given by

\[
\Sigma_{00} : \quad \dot{x}(t) = f(t, x(t), h(t, x), 0), \quad x(0) = \zeta(0),
\]

is called the reduced system, and the system given by

\[
\Sigma_b : \quad \frac{d\nu}{dt} = g(t, x, \nu + h(t, x), 0), \quad \nu(0) = \eta_0 - h(t, \zeta_0)
\]

(17)

is called the boundary layer system, where \( \eta_0 = \eta(0) \) and \( \zeta_0 = \zeta(0) \) are fixed initial parameters, and \( (t, x) \in [0, \infty) \times D_x \). The new time scale \( \tau \) is related to the original one via \( \tau = \frac{1}{\alpha} \). The next theorem is due to Tikhonov.

**Theorem 4.1:** Consider the singular perturbation system \( \Sigma_0 \) given in (14) and let \( u = h(t, x) \) be an isolated root of (15). Assume that the following conditions are satisfied for all \( (t, x, u - h(t, x), \epsilon) \) in \( [0, \infty) \times D_x \times D_u \times [0, \epsilon] \) for some domains \( D_x \subset \mathbb{R}^n \) and \( D_u \subset \mathbb{R}^m \), which contain their respective origins:

**A1.** On any compact subset of \( D_x \times D_u \), the functions \( f, g \), their first partial derivatives with respect to \( (x, u, \epsilon) \), and the first partial derivative of \( g \) with respect to \( t \) are continuous and bounded, \( h(t, x) \) and \( \frac{\partial g}{\partial t}(t, x, u, 0) \) have bounded first derivatives with respect to their
arguments, \( \frac{\partial V}{\partial x} \left( t, x, h(t, x) \right) \) is Lipschitz in \( x \), uniformly in \( t \), and the initial data given by \( \xi \) and \( \eta \) are smooth functions of \( \epsilon \).

A2. The origin is an exponentially stable equilibrium point of the reduced system \( \Sigma_{00} \) given by equation (16). There exists a Lyapunov function \( V : [0, \infty) \times D_x \to [0, \infty) \) that satisfies

\[
W_1(x) \leq V(t, x) \leq W_2(x)
\]

\[
\frac{\partial V}{\partial t} (t, x) + \frac{\partial V}{\partial x} (t, x) f(t, x, h(t, x), 0) \leq -W_3(x)
\]

for all \( (t, x) \in [0, \infty) \times D_x \), where \( W_1, W_2, W_3 \) are continuous positive definite functions on \( D_x \), and let \( c \) be a positive constant such that \( \{x \in D_x \mid W_3(x) \leq c\} \) is a compact subset of \( D_x \).

A3. The origin is an equilibrium point of the boundary layer system \( \Sigma_0 \) given by equation (17), which is exponentially stable uniformly in \( (t, x) \).

Let \( R_0 \subset D_v \) denote the region of attraction of the autonomous system \( \frac{dv}{dt} = g(0, \xi_0, v + h(0, \xi_0), 0) \), and let \( \Omega_v \) be a compact subset of \( R_0 \). Then for each compact set \( \Omega_x \subset \{x \in D_x \mid W_2(x) \leq \rho c, 0 < \rho < 1\} \), there exists a positive constant \( \epsilon_0 \) such that for all \( t \geq 0, \xi_0 \in \Omega_x, \eta_0 = h(0, \xi_0) \in \Omega_v \), and \( 0 < \epsilon < \epsilon_0 \), \( \Sigma_0 \) has a unique solution \( x_{\epsilon} \) on \( [0, \infty) \) and

\[
x_{\epsilon}(t) - x_{00}(t) = O(\epsilon)
\]

holds uniformly for \( t \in [0, \infty) \), where \( x_{00}(t) \) denotes the solution of the reduced system \( \Sigma_{00} \) in (16).

We will refer to the following Remark which will be used to prove exponential stability of the boundary layer system.

**Remark 1:** Verification of Assumption A3 can be done via a Lyapunov argument: if there is a Lyapunov function \( \Sigma : [0, \infty) \times D_x \times D_v \), that satisfies

\[
c_1 \|v\|^2 \leq V(t, x, v) \leq c_2 \|v\|^2
\]

\[
\frac{\partial V}{\partial v} g(t, x, v + h(t, x)) \leq -c_3 \|v\|^2,
\]

for all \( (t, x, v) \in [0, \infty) \times D_x \times D_v \), then Assumption A3 is satisfied.

**B. Convergence Result**

The condition in (4) implies existence of an isolated root for \( f(t, e, u) = 0 \). Let \( u = h(t, e) \) denote this isolated root. Following *Theorem 4.1*, the reduced system for (7), (9) is given by

\[
e \frac{d\nu}{dt} = A_r e(t), \quad e(0) = e_0,
\]

and the boundary layer system is given by

\[
\frac{dv}{dt} = \Lambda(t, e, \nu + h(t, e)) Q^{-1}(t, e) f(t, e, \nu + h(t, e)),
\]

with \( \nu(0) = \nu_0 \).

The main theorem of this paper is stated as follows:

**Theorem 4.2:** Subject to the condition in (4), the origin of (21) is exponentially stable. Moreover, there exists an \( \epsilon_* > 0 \) and a \( T > 0 \), such that for all \( t \geq T \) and \( 0 < \epsilon < \epsilon_* \),

the system given by (3), (9) has a unique solution on \( [0, \infty) \), \( x_r(t) = x_r(t) + O(\epsilon) \) holds uniformly for \( t \in [T, \infty) \).

**Proof.** We will verify that the assumptions in Tikhonov’s theorem are satisfied. Verification of A1 is straightforward, since the condition in (4) implies that for any compact subset, the function \( f \) and its first partial derivatives with respect to \( t \) are continuous and bounded, \( h(t, e) \) and \( \frac{\partial f}{\partial e} (t, e, \nu + h(t, e)) \) have bounded first derivatives with respect to their arguments, and \( \frac{\partial f}{\partial \nu} (t, e, \nu + h(t, e)) \) is Lipschitz in \( e \), uniformly in \( t \). Since \( A_r \) is Hurwitz by definition, the origin is an exponentially stable equilibrium point for the reduced system given in (20), and, hence, A2 is satisfied.

For verification of A3, consider the following Lyapunov function candidate for the boundary layer system (21):

\[
V(t, e, \nu)(\tau) = (Q^{-1}(t, e) f(t, e, \nu + h(t, e)))^T (Q^{-1}(t, e) f(t, e, \nu + h(t, e))) + \Lambda(t, e, \nu + h(t, e)),
\]

(22)

Since \( f(t, e, h(t, e)) = 0 \), then \( Q^{-1}(t, e) f(t, e, h(t, e)) = 0 \), and hence \( V(t, e, 0) = 0 \). Therefore

\[
V(t, e, \nu) = \int_L \nabla V(t, e, s) ds,
\]

(23)

where \( \int_L \) assumes integration along the pass \( L \) from zero to \( \nu \), and \( \nabla V(t, e, \nu) = (Q^{-1}(t, e) f(t, e, \nu + h(t, e)))^T \Lambda(t, e, \nu + h(t, e)). \)

Since

\[
\frac{\partial f(t, e, \nu + h(t, e))}{\partial \nu} = Q(t, e) \Lambda(t, e, \nu + h(t, e)),
\]

then \( f(t, e, \nu + h(t, e)) = \int_L Q(t, e) \Lambda(t, e, s + h(t, e)) ds \). Hence, it follows from (23) that

\[
V(t, e, \nu) = \int_L \int_{s_1} (Q(t, e) \Lambda(t, e, s + h(t, e)) ds_1)^T 

\]

(24)

and consequently,

\[
\Lambda(t, e, s + h(t, e)) ds,
\]

(25)

where \( \int_L, \int_{s_1} \) assume integration along the pass \( L \) and \( L_1 \) from zero to \( s \), respectively. From (4), (12), one can derive that \( \lambda_\min(\Lambda(T, t, s_1 + h(t, e)) \Lambda(t, e, s + h(t, e))) \geq \min(\beta_1^2, \beta_2^2) \), where \( \lambda_\min(\cdot) \) denotes the minimum eigenvalue. Consequently,

\[
V(t, e, \nu) \geq \int_L \int_{s_1} \lambda_\min(\Lambda(T, t, s_1 + h(t, e))) \Lambda(T, t, s + h(t, e)) ds_1 ds \int_0^\infty \lambda_\min(\Lambda(T, t, s_1 + h(t, e))) ds_1 ds_2 d\xi.
\]

(26)

Therefore,

\[
V(t, e, \nu) \geq c_1 \|\nu\|^2.
\]

(27)
Condition A1 implies that there exists $c_2$ such that $V(t,e,\nu) \leq c_2||\nu||^2$, which verifies (18). Using the relationships in (21) and (22), one obtains

\[
\dot{V}(t,e,\nu) = (Q^{-1}(t,e)f(t,e,\nu+h(t,e)))^T \\
\Lambda(t,e,\nu+h(t,e))\dot{V}(t,e,\nu+h(t,e)) \\
Q^{-1}(t,e)f(t,e,\nu+h(t,e))
\]

\[
\text{(28)}
\]

which implies that $\dot{V}(t,e,\nu) \leq -2c_1V(t,e,\nu)$. Consequently, the inequality in (27) leads to $\dot{V}(t,e,\nu) \leq -2c_2||\nu||^2$, and this proves that (19) holds with $c_3 = 2c_2$. Hence, A3 holds, and the boundary layer (21) has exponentially stable origin. Following Tikhonov’s theorem, there exists an $\epsilon_\ast > 0$ and a $T > 0$, such that for all $t \geq T$ and $0 < \epsilon < \epsilon_\ast$, the system given by (3), (9) has a unique solution, $x_\epsilon(t)$ on $[0, \infty)$, and $x_\epsilon(t) = x_r(t) + O(\epsilon)$ holds uniformly for $t \in [T, \infty)$.

V. Simulations

For simplicity, let $m_1 = m_2 = 1\text{ kg}$, $l_1 = l_2 = 1\text{ m}$, and let $g = 9.81\text{ m/s}^2$. Since actuator models are often represented by hyperbolic tangent functions of control input $u$, we assume that $M_1$ and $M_2$ are given as

\[
M_1(u_1) = \tanh(u_1) + 0.35u_1 \\
M_2(u_2) = \tanh(u_2) + 0.25u_2,
\]

so that

\[
\frac{\partial M_1(u_1)}{\partial u_1} \geq 0.35 > 0,
\]

\[
\frac{\partial M_2(u_2)}{\partial u_2} \geq 0.25 > 0.
\]

From (13) it follows that the matrices $Q(x)$ and $\Lambda(u)$ are given by

\[
Q(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) \\ f_{21}(x) & f_{22}(x) \end{bmatrix},
\]

\[
\Lambda(u) = \begin{bmatrix} 1.35 - \tanh^2(u_1) & 0 \\ 0 & 1.25 - \tanh^2(u_2) \end{bmatrix}.
\]

We consider the following reference function:

\[
r_d(t) = \begin{cases} \sin(t) + e^{-t}, \\ \sin(2t) + e^{-t}. \end{cases}
\]

This parametrization ensures that as the exponential term decays, $x_1$ and $x_2$ follow a figure $\infty$ pattern as described in [20]. Let

\[
A_r = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -20 & -9 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -6 & -5 \end{bmatrix}, B_r = \begin{bmatrix} 0 & 0 \\ 20 & 0 \\ 0 & 0 \\ 0 & 6 \end{bmatrix}.
\]

In order to ensure asymptotic convergence of the states $x_1(t)$ and $x_3(t)$ to the reference input, we define $r(t)$ as

\[
r(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \dot{r}_d(t) + \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix} \ddot{r}_d(t) + \begin{bmatrix} 20 & 0 \\ 0 & 5 \end{bmatrix} \dddot{r}_d(t).
\]

The fast dynamics are designed as:

\[
0.003\dot{x}(t) = -\Lambda(u(t))(Q^{-1}(t,e))^T f(t,e,u(t)),
\]

\[
u_0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T,
\]

where

\[
f(t,e,u) = \begin{bmatrix} f_1(e + x^r(t),u) + 20(e_1 + x_1^r(t)) + 9(e_2 + x_2^r(t)) - 20(r_{d1}(t) + \frac{9}{20}\ddot{r}_{d1}(t)) - \ddot{r}_{d1}(t) \\ f_2(e + x^r(t),u) + 6(e_3 + x_3^r(t)) + 5(e_4 + x_4^r(t)) - 6(r_{d2}(t) + \frac{5}{6}\ddot{r}_{d2}(t)) - \ddot{r}_{d2}(t) \end{bmatrix}.
\]

The initial values used for this numerical simulation are:

\[
x_0 = [0 \ 0 \ 0 \ 0]^T \text{ and } x_{0}^r = [0 \ 0 \ 0 \ 0]^T.
\]

The plots in Figures 2, 3 show the simulation results. Figure 2 shows the closed-loop tracking performance of the reference state $x_1(t)$ to the desired reference trajectory $r_d(t)$ and the closed-loop tracking performance of the reference state $x_r(t)$ by system state $x(t)$, while Figure 3 shows the phase plot of the states $x_1(t)$ and $x_2(t)$ and the control effort $u(t)$. Note that all signals are bounded and $x(t)$ tracks $r_d(t)$.

![Reference State Tracking of Reference Input](image-url)

![System State Tracking of Reference State](image-url)
VI. CONCLUSIONS

In this paper, using tools from singular perturbations theory, we have solved the tracking problem for the nonaffine pendulum dynamics. An approximate dynamic inversion control method was developed that achieved time-scale separation between the system dynamics and the controller dynamics. This was verified using Tikhonov’s theorem from singular perturbations theory.

REFERENCES