Verifiable Adaptive Flight Control: UCAV and Aerial Refueling

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This paper presents application of $L_1$ adaptive controller to two benchmark flight control applications. The benefit of the proposed adaptive control approach is its promise for development of theoretically justified tools for Verification and Validation (V&V) of adaptive systems. It has apriori predictable performance bounds and guaranteed, bounded away from zero, time-delay margin in the presence of fast adaptation. Two flight control examples, Unmanned Combat Aerial Vehicle (UCAV) and Aerial Refueling Autopilot, are considered in the presence of nonlinear uncertainties and control surface failures. The $L_1$ adaptive controller without any redesign leads to scaled response for system’s both signals, input and output, dependent upon changes in the initial conditions, system parameters and uncertainties.

I. Introduction

Since the early 1990’s, Air Force, Navy and NASA, in collaboration with industry and academia have made significant progress towards maturing the adaptive control theory for application to reconfigurable/damage adaptive flight control for aircraft and weapon systems. Reconfigurable flight control refers to the ability of a flight control system to adapt to unknown failures, damage, and uncertain aerodynamics. Flight control systems that are adaptive

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and reconfigurable constitute an important element in the design of mission effective unmanned combat systems. These properties of the control system can increase the reliability of unmanned systems.

Both indirect and direct adaptive control methods have been investigated, and several approaches have been successfully flown on manned and unmanned aircraft, and also on advanced weapon systems. Indirect Adaptive based approaches include Self-Repairing Flight Control Systems (SRFCS), Self Designing Controller (SDC) and Intelligent Flight Control System (IFCS) Generation I. Direct adaptive control based approaches have been successfully used in Reconfigurable Control for Tailless Fighters (RESTORE) program. The application of those technologies under the RESTORE program led to the flight testing of the approach on the Boeing/NASA X-36 Agility Research Test Aircraft. Success in the RESTORE dynamic inversion based adaptive neural-network control of aircraft and missiles offered the potential to develop flight control systems without knowledge of the aerodynamics. These technological advances, paved the road to successful transition of adaptive flight control methods into two Boeing / USAF production programs [1–4].

Some previous work done by the authors [5–8] demonstrates the benefits and the potential of direct adaptive control for flying vehicles, and also summarizes the challenges and the open problems in this direction [9]. The recently developed $L_1$ adaptive controller addressed some of those challenges by establishing a new paradigm for design of adaptive control systems. The $L_1$ adaptive control methodology extends the classical notions of gain and phase margins to a nonlinear adaptive control scheme, and, for the first time, enables the design of a nonlinear closed-loop control system with guaranteed, analytically provable, bounded away from zero, time-delay margin [10, 11]. Its guaranteed performance bounds and systematic design procedures can significantly reduce the tuning effort required for achieving desired closed-loop performance, while operating in the presence of uncertainties. The performance limitations of $L_1$ adaptive controller are shown to be consistent with hardware limitations. Furthermore, its ability of fast adaptation allows for control of nonlinear systems by adapting two parameters only [15]. This, in particular, eliminates the need for selecting and tuning neural network basis functions, and with that significantly reduces the control design efforts. These features lay foundation for development of theoretically justified tools for Validation and Verification (V&V) of adaptive systems.

We present the $L_1$ adaptive controller for two benchmark examples of flight control. First, the X-45A Unmanned Combat Aerial Vehicle in the presence of unknown actuator failures and pitching moment uncertainty is discussed. Simulation results demonstrate the benefits of the proposed $L_1$ adaptive controller. The time-delay margin is computed and shown to be consistent with the theoretical predictions. Next, the Automated Aerial Refueling (AAR) problem is considered in the presence of vortex induced uncertainties. The simulation is done using Barron’s Associates tailless aircraft model [14]. In both applications, a single design of $L_1$ adaptive controller achieves (guaranteed) scaled response for system’s both signals, dependent upon changes in initial conditions, reference inputs and system uncertainties.
II. Problem Formulation

The adaptive control architecture includes a fixed robust baseline controller and an augmentation of it by an adaptive controller. The baseline controller is designed to yield consistent nominal system performance in the absence of failures, while the adaptive element provides adaptation and reconfiguration in the presence of system uncertainties (e.g., battle damage), control failures and unknown aerodynamics. The linearized open loop plant dynamics can be generalized and written in the form:

\[
\begin{aligned}
\dot{x}_p(t) &= A_p x_p(t) + B_p \Lambda [u(t) + K_0(t, x_p(t))] , \quad x_p(0) = x_{p0} \\
y(t) &= C_p x_p(t) + D_p \Lambda [u(t) + K_0(t, x_p(t))] \\
z_p(t) &= F y(t)
\end{aligned}
\]

where \( A_p \in \mathbb{R}^{n_1 \times n_1}, \ B_p \in \mathbb{R}^{n_1 \times m}, \ C_p \in \mathbb{R}^{l \times n_1}, \ D_p \in \mathbb{R}^{l \times m} \) and \( F \in \mathbb{R}^{p \times l} \) are known matrices, \( u(t) \in \mathbb{R}^m \) is virtual control input, \( \Lambda \in \mathbb{R}^{m \times m} \) is unknown with strictly positive diagonal element \( \Lambda_i, \ i = 1, 2, \cdots, m, \) and represents control effectiveness uncertainties, \( K_0(t, x_p) \in \mathbb{R}^m \) is unknown nonlinear function of system states. The output \( y(t) \) represents the sensor measurements, and \( z_p(t) \) is the subset of the plant outputs that are to be controlled.

The dynamics of the baseline controller can be generalized and written in the form:

\[
\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_{1c} z_c(t) + B_{2c} z_p(t) \\
u(t) &= C_c x_c(t) + D_{1c} z_c(t) + D_{2c} z_p(t)
\end{aligned}
\]

where \( z_c(t) \in \mathbb{R}^q \) is the vector of outer-loop commands from guidance, and \( x_c \in \mathbb{R}^{n_2} \). Substituting \( z_p(t) \) from (1) into (2), yields:

\[
\begin{aligned}
\dot{x}_c(t) &= A_c x_c(t) + B_{1c} z_c(t) + B_{2c} F [C_p x_p(t) + D_p \Lambda (u(t) + K_0(t, x_p(t)))] \\
u(t) &= C_c x_c(t) + D_{1c} z_c(t) + D_{2c} F [C_p x_p(t) + D_p \Lambda (u(t) + K_0(t, x_p(t)))]
\end{aligned}
\]

Consequently, the second equation in (3) can be solved explicitly for \( u(t) \):

\[
\begin{aligned}
u(t) &= (\mathbb{I} - D_{2c} F D_p \Lambda)^{-1} [C_c x_c(t) + D_{1c} z_c(t) + D_{2c} F C_p x_p(t) + D_{2c} F D_p \Lambda K_0(t, x_p(t))] \\
&= (\mathbb{I} - D_{2c} F D_p \Lambda)^{-1} [D_{2c} F C_p \ C_c] \begin{bmatrix} x_p(t) \\ x_c(t) \end{bmatrix} + (\mathbb{I} - D_{2c} F D_p \Lambda)^{-1} D_{1c} z_c(t) \\
&+ (\mathbb{I} - D_{2c} F D_p \Lambda)^{-1} D_{2c} F D_p \Lambda K_0(t, x_p(t)).
\end{aligned}
\]

The baseline controller \( u_L(t) \), assuming no uncertainties, \( \Lambda = \mathbb{I}, \ K_0(t, x_p(t)) = 0 \), can be written as:

\[
u_L(t) = K_x^T x(t) + K_z^T z_c(t)
\]
where
\[ K_x^T = (1 - D_2cFD_p)^{-1} [D_2cFC_p \quad C_c] , \quad K_z^T = (1 - D_2cFD_p)^{-1} D_{1c} . \]

The extended system dynamics are
\[
\begin{bmatrix}
\dot{x}_p(t) \\
\dot{x}_c(t) \\
y(t)
\end{bmatrix} =
\begin{bmatrix}
A_p & 0 & \Lambda [u(t) + K_0(t,x_p(t))] + B_c z_c(t) \\
B_2cFC_p & A_c & 0 \\
C_p & 0 & C)
\end{bmatrix}
\begin{bmatrix}
x_p(t) \\
x_c(t) \\
x(t)
\end{bmatrix}
\begin{bmatrix}
B_p \\
B_{2c}FD_p \\
B_1
\end{bmatrix}
\begin{bmatrix}
0 \\
x(t)
\end{bmatrix}
\]

or equivalently,
\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 \Lambda [u(t) + K_0(t,x_p(t))] + B_2 z_c(t) \\
y(t) &= Cx(t) + D_p \Lambda [u(t) + K_0(t,x_p(t))] ,
\end{align*}
\]

where \( x \in \mathbb{R}^n, n = n_1 + n_2. \)

In the absence of uncertainties, the linear closed-loop dynamics take the form:
\[
\begin{align*}
\dot{x}_m(t) &= (A + B_1 K_x^T) x_m(t) + (B_1 K_z^T + B_2) z_c(t) \\
y_m(t) &= (C + D_p K_x^T) x_m(t) + D_z K_z^T z_c(t)
\end{align*}
\]

The closed-loop system (8) defines the desired nominal response, with \( A_m \) being Hurwitz. The control design relies on the following assumptions.

**Assumption 1**
\[
\Lambda_i \in [\Lambda_l, \Lambda_u], \quad i = 1, 2, \cdots, m,
\]
where \( \Lambda_u > \Lambda_l > 0 \) are known.

Let \( f(t,x(t)) = \Lambda K_0(t,x_p(t)) \), where \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m \) is an unknown nonlinear function. Notice that
\[
\begin{align*}
\|x_p\|_\infty &\leq \|x\|_\infty \\
\|x_{p1} - x_{p2}\|_\infty &\leq \|x_1 - x_2\|_\infty .
\end{align*}
\]

**Assumption 2 (Semiglobal Lipschitz Condition)** For any \( \delta > 0 \), there exist positive \( L_\delta \) and \( B \) such that
\[
\begin{align*}
\|f(t,x_1) - f(t,x_2)\|_\infty &\leq L_\delta \|x_1 - x_2\|_\infty . \quad (10) \\
\|f(t,0)\|_\infty &\leq B , \quad (11)
\end{align*}
\]

for all \( \|x_1\|_\infty \leq \delta \) and \( \|x_2\|_\infty \leq \delta \) uniformly in \( t \geq 0. \)
Assumption 3 (Semiglobal uniform boundedness of partial derivatives) For any \( \delta > 0 \), there exist \( d_{f,x}(\delta) > 0 \), and \( d_{f,t}(\delta) > 0 \) such that for any \( \|x\|_\infty \leq \delta \), the partial derivatives of \( f(t, x) \) are piece-wise continuous and bounded

\[
\left\| \frac{\partial f(t, x)}{\partial x} \right\|_\infty \leq d_{f,x}(\delta), \quad \left\| \frac{\partial f(t, x)}{\partial t} \right\|_\infty \leq d_{f,t}(\delta).
\]

The control objective is to design state feedback control \( u(t) \) such that despite the system uncertainties \( \Lambda \) and \( K_0(t, x_p(t)) \), all closed loop signals are bounded and the system state \( x(t) \) tracks the desired model state \( x_m(t) \). Next we present the \( \mathcal{L}_1 \) adaptive controller that achieves this objective without the need for any retuning.

### III. \( \mathcal{L}_1 \) Adaptive Control

Consider the following control law

\[
u(t) = u_L(t) + u_{ad}(t), \quad (13)
\]

where \( u_L(t) \) is the component of the baseline linear controller, and \( u_{ad}(t) \) is the adaptive increment. The baseline controller is defined in (5), which is repeated here:

\[
u_L(t) = K_x^T x(t) + K_z^T z_c(t), \quad (14)
\]

where \( K_x \) and \( K_z \) denote the \((n \times m)\) and \((q \times m)\) nominal feedback and feedforward gain matrices, correspondingly. Using these stabilizing gains for the system in (6), it takes the form:

\[
\dot{x}(t) = (A + B_1 \Lambda K_x^T) x(t) + B_1 \Lambda (u_{ad} + K_0(t, x_p(t))) + (B_2 + B_1 \Lambda K_z^T) z_c(t). \quad (15)
\]

From (8) and (15), we have

\[
\dot{x}(t) = A_m x(t) + B_m z_c(t) + B_1 \left( \Lambda u_{ad}(t) + \Lambda K_0(t, x_p(t)) \right) + k_x^T x(t) + k_z^T z_c(t), \quad (16)
\]

where

\[
k_x^T = (\Lambda - I) K_x^T, \quad k_z^T = (\Lambda - I) K_z^T. \quad (17)
\]

**Remark 1** In the absence of actuator failures, i.e. when \( \Lambda = I_m \), we have \( k_x(t) = 0_{n \times m} \) and \( k_z(t) = 0_{q \times m} \). This implies that the adaptive system augments the baseline inner-loop controller, and therefore the incremental adaptive feedback gains can be initialized at zero.

The design of \( \mathcal{L}_1 \) adaptive controller involves a diagonal transfer function matrix \( D(s) \) with strictly proper transfer function elements and a positive diagonal feedback gain matrix \( k \in \mathbb{R}^{m \times m} \)

\[
D(s) = \begin{bmatrix} D_1(s) & 0 & 0 & \cdots & 0 \\ 0 & D_2(s) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & D_m(s) \end{bmatrix} \quad (18)
\]
Furthermore, $D(s)$ and $k$ lead to strictly proper stable

$$C(s) = \begin{bmatrix} C_1(s) & 0 & 0 & \cdots & 0 \\ 0 & C_2(s) & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & C_m(s) \end{bmatrix},$$

with low-pass gain $C_i(0) = 1$. One simple choice is

$$D_i(s) = \frac{1}{s},$$

which yields a first order strictly proper $C(s)$ in the following form

$$C(s) = \begin{bmatrix} \frac{\Lambda_1 k_1}{s + \Lambda_1 k_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{\Lambda_2 k_2}{s + \Lambda_2 k_2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{\Lambda_m k_m}{s + \Lambda_m k_m} \end{bmatrix}. \tag{22}$$

Further, let

$$L_x = \max\{\|\Delta_l - 1\|, |\Delta_u - 1|\}\|K_x^T\|_1 = \max\{\|\Delta_l - 1\|, |\Delta_u - 1|\} \max_{i=1,2,\ldots,m} \left(\sum_{j=1}^n |K_{xji}|\right), \tag{23}$$

$$L_z = \max\{\|\Delta_l - 1\|, |\Delta_u - 1|\}\|K_z^T\|_1 = \max\{\|\Delta_l - 1\|, |\Delta_u - 1|\} \max_{i=1,2,\ldots,m} \left(\sum_{j=1}^n |K_{zji}|\right),$$

where $K_{xji}(t)$ are the $j^{th}$ row $i^{th}$ column element of $K_x$. We now state the $\mathcal{L}_1$ performance requirement that ensures stability of the entire system and desired transient performance, as in [15].

Let

$$H(s) = (sI - A_m)^{-1}B_1, \tag{24}$$

$$H_m(s) = (sI - A_m)^{-1}B_m, \tag{25}$$
and $r_0(t)$ be the signal with its Laplace transformation $(sI - A_m)^{-1}x_0$. Since $A_m$ is Hurwitz and $x_0$ is finite, $\|r_0\|_{\mathcal{L}_\infty}$ is finite.

**$\mathcal{L}_1$-gain stability requirement:** The choice of $D(s)$ and $k$ needs to ensure that there exists $\rho_r$ such that:

$$\|G(s)\|_{\mathcal{L}_1} < \left(\rho_r - (\|G(s)\|_{\mathcal{L}_1}L_z + \|H_m(s)\|_{\mathcal{L}_1} \|z_c\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty})/(\rho_rL_g + \rho_rL_x + B)\right),$$  \hspace{1cm} (26)

where

$$G(s) = H(s)(I_{m \times m} - C(s)).$$

**Remark 2** We notice that the $\mathcal{L}_1$-gain stability requirement in (26) is a consequence of the semiglobal Lipschitz property of $f(t, x)$, stated in Assumption 1. If $f(t, x)$ is globally Lipschitz with uniform Lipschitz constant $L_g$, then

$$\lim_{\rho_r \to \infty} \left(\rho_r - (\|G(s)\|_{\mathcal{L}_1}L_z + \|H_m(s)\|_{\mathcal{L}_1} \|z_c\|_{\mathcal{L}_\infty} - \|r_0\|_{\mathcal{L}_\infty})/(\rho_rL_g + \rho_rL_x + B)\right) = \frac{1}{L_g + L_x} = \frac{1}{L},$$

where $L = L_g + L_x$, and the $\mathcal{L}_1$-gain stability requirement in (26) degenerates into

$$\|G(s)\|_{\mathcal{L}_1} < 1/L,$$

which is the same as the one derived in [16] for systems with constant unknown parameters. Notice that (26) is a sufficient condition for stability.

The elements of $\mathcal{L}_1$ adaptive controller are introduced next:

**State Predictor:** We consider the following state predictor:

$$\dot{x}(t) = A_m \tilde{x}(t) + B_m z_c(t) + B_1 \left(\tilde{\Lambda}(t) u_{ad}(t) + \tilde{\theta}(t) \|x(t)\|_{\mathcal{L}_\infty} + \tilde{\sigma}(t) + \tilde{k}_x(t) x(t) + \tilde{k}_z(t) z_c(t)\right),$$

$$\dot{y}(t) = c^T \tilde{x}(t), \quad \tilde{x}(0) = x_0,$$

(27)

where $\tilde{\theta}(t) \in \mathbb{R}^{1 \times m}$ and $\tilde{\sigma}(t) \in \mathbb{R}^{1 \times m}$.

**Adaptive Laws:** Adaptive estimates $\hat{\Lambda}(t), \hat{\theta}(t), \hat{\sigma}(t), \hat{k}_x(t), \hat{k}_z(t)$ are governed by the following adaptation laws.

$$\dot{\hat{\theta}}(t) = \Gamma \text{Proj}(\hat{\theta}(t), -\|x(t)\|_{\mathcal{L}_\infty} \tilde{x}^T(t) PB_1), \quad \hat{\theta}(0) = \hat{\theta}_0,$$

$$\dot{\hat{\sigma}}(t) = \Gamma \text{Proj}(\hat{\sigma}(t), -\tilde{x}^T(t) PB_1), \quad \hat{\sigma}(0) = \hat{\sigma}_0,$$

$$\dot{\hat{\Lambda}}(t) = \Gamma \text{Proj}(\hat{\Lambda}(t), -u_{ad}(t) \tilde{x}^T(t) PB_1), \quad \hat{\Lambda}(0) = \hat{\Lambda}_0,$$

$$\dot{\hat{k}_x}(t) = \Gamma \text{Proj}(\hat{k}_x(t), -x(t) \tilde{x}^T(t) PB_1), \quad \hat{k}_x(0) = 0,$$

$$\dot{\hat{k}_z}(t) = \Gamma \text{Proj}(\hat{k}_z(t), -z_c(t) \tilde{x}^T(t) PB_1), \quad \hat{k}_z(0) = 0,$$

where $\tilde{x}(t) = \tilde{x}(t) - x(t), \Gamma = \Gamma_v I$ is the adaptation gain matrix, $P$ is the solution of the algebraic equation $A_m^T P + P A_m = -Q, Q > 0$, and the projection operator ensures that the adaptive estimates $\hat{\Lambda}(t), \hat{\theta}(t), \hat{\sigma}(t), \hat{k}_x(t), \hat{k}_z(t)$ remain inside the compact sets $[\omega_l, \omega_u], [-\theta_b, \theta_b], [-\sigma_b, \sigma_b], [-L_x, L_x], [-L_z, L_z]$ respectively, with $\theta_b, \sigma_b$ defined as follows

$$\theta_b = L_\rho, \quad \sigma_b = B + \epsilon,$$

(28)
where $\epsilon$ is an arbitrary positive constant and
\[ \rho = \rho_r + \beta, \tag{29} \]
with $\beta$ being an arbitrary positive constant as well.

**Control Law:** The control signal is generated through gain feedback of the following system:
\[ \chi(s) = D(s)\bar{r}(s), \quad u_{ad}(s) = -k\chi(s), \tag{30} \]
where $\bar{r}(s)$ is the Laplace transformation of
\[ \bar{r}(t) = \hat{\Lambda}(t)u_{ad}(t) + \hat{\theta}^\top(t)x(t)\|x(t)\|_\infty + \hat{\sigma}^\top(t) + \hat{k}_x^\top(t)x(t) + \hat{k}_z^\top(t)z_c(t). \tag{31} \]

The complete $L_1$ adaptive controller consists of (27), (28) and (30) subject to the $L_1$-gain stability requirement in (26).

**IV. Analysis of $L_1$ Adaptive Controller**

**IV.A. Reference System**

We now consider the following closed-loop reference system with its control signal and system response being defined as follows:
\[ \dot{x}_{ref}(t) = A_m x_{ref}(t) + B_m z_c(t) + B_1\left(f(t,x_{ref}(t)) + \Lambda u_{ref}(t) + k_x^\top x_{ref} + k_z^\top z_c(t)\right) \tag{32} \]
\[ u_{ref}(s) = -\Lambda^{-1}C(s)\bar{r}_{ref}(s) \tag{33} \]
\[ y_{ref}(t) = c^\top x_{ref}(t), \quad x_{ref}(0) = x_0 \tag{34} \]
where $\bar{r}_{ref}(s)$ is the Laplace transformation of the signal $\bar{r}_{ref}(t) = f(t,x_{ref}(t)) + k_x^\top x_{ref}(t) + k_z^\top z_c(t)$. The next Lemma establishes the stability of the closed-loop system in (32)-(34).

**Lemma 1** For the closed-loop reference system in (32)-(34) subject to the $L_1$-gain stability condition in (26), we have
\[ \|x_{ref}\|_{L_\infty} < \rho_r, \tag{35} \]
\[ \|u_{ref}\|_{L_\infty} < \rho_{u_r}, \tag{36} \]
where $\rho_r$ is defined in (26) and
\[ \rho_{u_r} = \|\Lambda^{-1}C(s)\|_{L_1}(L_{\rho_r}\rho_r + L_x\rho_r + B + L_z\|z_c\|_{L_\infty}). \]
A proof can be found in [15].
IV.B. Equivalent Linear Time-Varying System

In this section we demonstrate that the nonlinear system in (16) can be transformed into a linear system with unknown time-varying parameters. We need to first introduce several notations. Choose $\gamma_0$ to satisfy
\[
\gamma_1 \triangleq \frac{\|C(s)\|_{L_1}}{1 - \|G(s)\|_{L_1}} \gamma_0 + \beta_1 < \beta,
\]
where $0 < \beta_1 < \beta$ is an arbitrary positive constant. We notice that $\gamma_0$ can be arbitrarily small since $\beta$ and $\beta_1$ can be set arbitrarily. We will prove that by increasing the adaptive gain, $\gamma_0$ can serve as an upper bound for the tracking error signal $\tilde{x}(t)$. Define
\[
B_r \triangleq [B_1 \bar{B}_1],
\]
\[
H_r(s) \triangleq (sI - A_m)^{-1}B_r,
\]
\[
C_r(s) \triangleq \begin{bmatrix} \Lambda^{-1}C(s) & 0 \\ 0 & \bar{C}(s) \end{bmatrix},
\]
where the choice of $\bar{B}_1$ renders $B_r \in \mathbb{R}^{n \times n}$ full rank, and $\bar{C}(s)$ is an arbitrary diagonal strictly proper stable transfer function. Further, let
\[
\rho_u = \rho_u + \gamma_2,
\]
\[
\gamma_2 = \|\Lambda^{-1}C(s)\|_{L_1} (L_\rho + L_x) \gamma_1 + \|\Lambda^{-1}C_rH_r^{-1}\|_{L_1} \gamma_0.
\]

Lemma 2 If
\[
\|x_t\|_{L_\infty} \leq \rho,
\]
\[
\|u_t\|_{L_\infty} \leq \rho_u,
\]
where $\|\cdot\|_{L_\infty}$ denotes the truncated $L_\infty$ norm (see Appendix for definition), there exist differentiable $\theta(\tau)$ and $\sigma(\tau)$ with bounded $\dot{\theta}(\tau)$ and $\dot{\sigma}(\tau)$ over $\tau \in [0, t]$ such that
\[
\|\theta(\tau)\|_{\infty} < \theta_b,
\]
\[
\|\sigma(\tau)\|_{\infty} < \sigma_b,
\]
\[
f(\tau, x(\tau)) = \theta(\tau)\|x(\tau)\|_{\infty} + \sigma(\tau).
\]

Refer to [15] for the proof.

Since
\[
\|x_0\|_{\infty} \leq \rho_r \leq \rho, \quad u(0) = 0,
\]
and $x(t), u(t)$ are continuous, there always exists $t$ such that
\[
\|x_t\|_{L_\infty} \leq \rho, \quad \|u_t\|_{L_\infty} \leq \rho_u.
\]
It follows from (47) and Lemma 2 that the system in (16) can be rewritten over \( [0, t] \) as:

\[
\dot{x}(\tau) = A_m x(\tau) + B_m z_c(\tau) + B_1 \left( \theta^T(\tau) \| x(\tau) \|_\infty + \Lambda u_{ad}(\tau) + \sigma(\tau) + k_x^T x(\tau) + k_z^T z_c(\tau) \right),
\]
\[
y(\tau) = c^T x(\tau), \quad x(0) = x_0,
\]

where \( \theta(\tau), \sigma(\tau) \) are unknown time-varying signals subject to:

\[
|\theta(\tau)| < \theta_b, \quad |\sigma(\tau)| < \sigma_b, \quad \forall \tau \in [0, t].\tag{49}
\]

\[
|\dot{\theta}(\tau)| \leq d_{\theta}(\rho, \rho_u), \quad |\dot{\sigma}(\tau)| \leq d_{\sigma}(\rho, \rho_u), \quad \forall \tau \in [0, t].\tag{50}
\]

**IV.C. Transient and Steady-State Performance**

**Theorem 1** Consider the reference system in (32)-(34) and the closed-loop \( \mathcal{L}_1 \) adaptive controller in (16), (27), (28), (30) subject to (26). If the adaptive gain is chosen to verify the lower bound:

\[
\Gamma_c > \frac{\theta_m(\rho, \rho_u)}{\lambda_{\min}(P)^{\gamma_0^2}},
\]

where

\[
\theta_m(\rho, \rho_u) \triangleq 4m^2 \theta_b^2 + 4m^2 \sigma_b^2 + m^2 (\Lambda_u - \Lambda_l)^2 + 4m^2 L_x^2 + 4m^2 L_z^2
\]

\[+ 4\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)} \left( d_{\theta}(\rho, \rho_u) + d_{\sigma}(\rho, \rho_u) \right),\tag{52}
\]

we have:

\[
\|\hat{x}\|_{\mathcal{L}_\infty} \leq \gamma_0,
\]
\[
\|x - x_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_1,
\]
\[
\|u - u_{\text{ref}}\|_{\mathcal{L}_\infty} < \gamma_2,
\]

where \( \gamma_1 \) and \( \gamma_2 \) are defined in (37) and (40).


It follows from (51) that we can achieve arbitrarily small \( \gamma_0 \) by increasing the adaptive gain.

**IV.D. Design Guidelines**

We note that the control law \( u_{\text{ref}}(t) \) in the closed-loop reference system, which is used in the analysis of \( \mathcal{L}_\infty \) norm bounds, is not implementable since its definition involves the unknown parameters. Theorem 1 ensures that the \( \mathcal{L}_1 \) adaptive controller approximates \( u_{\text{ref}}(t) \) both in transient and steady state. So, it is important to understand how these bounds can be used for ensuring uniform transient response with desired specifications. We notice that the following ideal control signal

\[
u_{id}(t) = \Lambda^{-1} \left( - f(t, x_{id}(t)) + k_x^T x_{id}(t) + k_z^T z_c(t) \right),\tag{56}
\]
is the one that leads to desired system response:

\[
\dot{x}_{id}(t) = A_m x_{id}(t) + B_m z_c(t), \quad y_{id}(t) = c^\top x_{id}(t)
\]  

(57)

by canceling the uncertainties exactly. In the closed-loop reference system (32)-(34), \(u_{id}(t)\) is further low-pass filtered by \(C(s)\) to have guaranteed low-frequency range. Thus, the reference system in (32)-(34) has a different response as compared to (57) achieved with (56). We refer the reader to [17] for specific design guidelines on selection of \(C(s)\) to ensure that the response of \(x_{ref}(t)\) and \(u_{ref}(t)\) can be made as close as possible to (57). Moreover, for systems linearly dependent upon unknown parameters, the (nonlinear) \(L_1\) adaptive controller has analytically computable time-delay margin [11]. The trade-off between the time-delay margin and the performance of the \(L_1\) adaptive controller depends solely upon the bandwidth of \(C(s)\). Increasing the bandwidth of \(C(s)\) leads to improved performance at the price of reduced time-delay margin. In [18], we consider constrained optimization of the performance and/or the robustness of \(L_1\) adaptive controller by resorting to appropriate Linear Matrix Inequality (LMI) type conditions. If the corresponding LMI has a solution, then arbitrary desired performance bound can be achieved, while retaining a prespecified lower-bound on the time-delay margin.

V. Unmanned Combat Aerial Vehicle

The \(L_1\) adaptive controller is applied to a model of the aerodynamically unstable X-45A Unmanned Combat Aerial Vehicle (UCAV) in the presence of actuator failures and pitch break uncertainty (unknown nonlinear in Angle of Attack (AoA) pitching moment increment). This uncertainty is introduced in pitch dynamics in order to model unknown changes in the aircraft pitching moment that can drive the aircraft into an uncontrollable (in AoA) region.

We recall the aircraft dynamics from (7):

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + B_1 \Lambda (u(t) + K_0(t, x_p(t))) + B_2 z_c(t), \\
x(0) &= x_0 = 0,
\end{align*}
\]  

(58)

where \(x(t) \in \mathbb{R}^9, u(t) \in \mathbb{R}^3\) (virtual control input), \(z_c(t) \in \mathbb{R}^4\) are the measured system states, control signals and reference inputs, respectively, \(A \in \mathbb{R}^{9 \times 9}, B_2 \in \mathbb{R}^{9 \times 4}, B_1 \in \mathbb{R}^{9 \times 3}\) are known matrices where the three columns of \(B_1\) are linearly independent,

\[
\Lambda = \begin{bmatrix}
\Lambda_1 & 0 & 0 \\
0 & \Lambda_2 & 0 \\
0 & 0 & \Lambda_3
\end{bmatrix}
\]  

(59)

is an unknown diagonal matrix with strictly positive diagonal elements. The state vector \(x = (\alpha, \beta, p, q, r, q_I, p_I, r_I, r_w)\) comprises five plant states \(x_p\), which include angle of attack \(\alpha\), angle of sideslip \(\beta\), body roll rate \(p\), body pitch rate \(q\), body yaw rate \(r\) and four baseline controller \((x_c)\) states, which include pitch, roll and yaw \((q_I, p_I \text{ and } r_I)\) integrator states and yaw rate washout filter signal \(r_w\). The vector \(z_c = (a_{z_{cmd}}^{cmd}, \beta_{cmd}, p_{cmd}, r_{cmd})\) consists of four inner loop
commands that include vertical acceleration, sideslip, roll rate and yaw rate, while $u(t)$ is the vector of virtual controls (roll, pitch and yaw control). Considering the control signal $u(t) = u_L(t) + u_{ad}(t)$, where $u_L(t)$ is defined in (5), the partially closed loop system from (16) is repeated here:

$$
\dot{x}(t) = A_m x(t) + B_m z_c(t) + B_1 \left( \Lambda u_{ad}(t) + \Lambda K_0(t, x_p(t)) + k_2^T x(t) + k_2^T z_c(t) \right).
$$

(60)

where the adaptive control signal $u_{ad}(t)$ is defined according to $L_1$ adaptive controller (27), (28) and (30), subject to the $L_1$-gain upper bound in (26). This is different from the approach, presented in [12], where RBF approximation was employed for approximation of the nonlinearity. This significantly reduces the design effort by eliminating the need for RBF distributions and centers/widths tuning.

Note that the vector-function $K_0(t, x_p)$ represents the matched unknown nonlinearities. In addition to unknown $K_0(t, x_p)$, $\Lambda$ models the loss of control effectiveness caused by actuator failures. The matrix $\Lambda$ models the uncertainties in virtual control $u(t)$, and is defined in a way that each diagonal element of $\Lambda$ represents a scaling factor for control effectiveness in particular channel. Thus, $\Lambda$ is diagonal and is different from the control allocation matrix, which has already been incorporated in $B_1$ in (58). The real control actuators are left outer board $\delta_{LOB}$, left middle board $\delta_{LMB}$, left inner board $\delta_{LIB}$, right inner board $\delta_{RIB}$, right middle board $\delta_{RMB}$, right outer board $\delta_{ROB}$ and thrust $\delta_{Tvec}$.

We only design the virtual controls (roll, pitch and yaw). When one or some of the control surfaces fail, the scaling factor for each virtual control will change. As long as the failures do not cause loss of controllability for each input, the matrix $\Lambda$ will remain diagonal and positive definite, as required by the main assumption of the above presented adaptive control approach.

The inner-loop control objective is to design a full state-feedback controller $u_{ad}(t)$ for (60) such that all closed-loop signals remain bounded, and the system state tracks the state of a desired reference model in the presence of uncertainties and actuator failures. We compare the performance of the nominal plant in the presence of pitch break and actuator failure with the performance of the closed-loop system with $L_1$ adaptive controller. For the $L_1$ adaptive controller we set $k_i = 20$, where $i = 1, 2, 3$, and $D(s) = \frac{1}{s}$ which verifies the $L_1$ gain stability condition, and we set $\Gamma = 100000$. The results are shown in Figures 1 and 2. For comparison purposes, simulation data are obtained from the following three closed-inner-loop systems: a) adaptation OFF, failures OFF, b) adaptation OFF, failures ON, c) d) $L_1$ adaptation ON, failures ON.

Figures 1-2 demonstrate the benefits of adaptation, when the right outboard (ROB) elevon fails at 1 second of the maneuver and the pitch break phenomenon is active throughout the entire maneuver. The failure of ROB causes the change of $\Lambda$ matrix:

$$
\Lambda = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0.1399 & 0 \\
0 & 0 & 0.3798
\end{bmatrix}.
$$

Figure 1 indicates that in spite of the unknown control failure and pitch break uncertainty, the $L_1$ adaptive system is able to quickly reconfigure and track the commanded vertical acceleration, sideslip angle, roll rate, and yaw rate.
signals, simultaneously. In fact, Figure 1 shows that with the adaptation turned on the desired/nominal system tracking behavior has been recovered. In addition, Figure 2 compares the three virtual control feedback signals, and confirms that the level of control activity is reasonable. In Figure 3, the subplot of Figure 1 is re-plotted to show the perfect tracking achieved by $L_1$ for vertical acceleration.

Figure 1. Inner-Loop Adaptation with ROB Elevon Failure and Pitch Break Phenomenon: Command Tracking

Figure 2. Inner-Loop Adaptation with ROB Elevon Failure and Pitch Break Phenomenon: Virtual Controls

When the reference commands change, the system output and input have scaled responses similar to those of linear
systems. These are shown in Figures 4, 5.

Next we discuss robustness of the scheme. The time-delay margin for the inner loop (without adaptive feedback) can be computed from the phase margin of the open loop transfer functions. The open loop transfer functions are calculated by breaking the virtual control ($\dot{p}, \dot{q}, \dot{r}$) command loops one at a time keeping the other two loops closed.
This is shown in Table 1.

Table 1. Phase and time-delay margins for the inner loop

<table>
<thead>
<tr>
<th>Loop Margin</th>
<th>( \dot{p} )</th>
<th>( \dot{q} )</th>
<th>( \dot{r} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Phase Margin (deg)</td>
<td>85.6</td>
<td>65.6</td>
<td>66.1</td>
</tr>
<tr>
<td>Cross-over Frequency (rad/sec)</td>
<td>4.25</td>
<td>6.01</td>
<td>4.27</td>
</tr>
<tr>
<td>Time-delay Margin (sec)</td>
<td>0.3515</td>
<td>0.1905</td>
<td>0.2702</td>
</tr>
</tbody>
</table>

To calculate the time-delay margin for the adaptive system, we introduce the time-delay at the plant input, and compute the margins using numerical simulations.

The time-delay margins for \( L_1 \) adaptive scheme are summarized in Table 2 for \( C(s) = \frac{1}{0.05s+1} \) and \( \Gamma = 100000 \). The worst case time-delay margin is 0.043 sec, which is small. Now we discuss how to improve the time-delay margin by the change of \( C(s) \) using the analysis from [11].

The time-delay margin for the \( L_1 \) adaptive scheme for systems linearly dependent upon unknown parameters (i.e., in the absence of nonlinear unknown function), can be computed analytically as \( \Gamma \rightarrow \infty \) [11]. In [11], \( C(s)/(1-C(s)) \) appears as a multiplier in the open-loop transfer function used for calculation of the lower bound on the time-delay margin for the \( L_1 \) adaptive scheme. It is obvious that one could choose \( C(s) \) judiciously to maximize the phase margin of the open-loop transfer function and minimize the cross-over frequency to obtain larger time-delay margin. Towards
Table 2. Time-delay margin for $L_1$ for $C(s) = \frac{1}{0.05s+1}$

<table>
<thead>
<tr>
<th>$\hat{p}$</th>
<th>$\hat{q}$</th>
<th>$\hat{r}$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>n/a</td>
<td>0.056</td>
<td>n/a</td>
<td>Individual</td>
</tr>
<tr>
<td>0.070</td>
<td>n/a</td>
<td>n/a</td>
<td>Individual</td>
</tr>
<tr>
<td>n/a</td>
<td>n/a</td>
<td>0.055</td>
<td>Individual</td>
</tr>
<tr>
<td>0.0430</td>
<td>0.0430</td>
<td>n/a</td>
<td>Two Loops</td>
</tr>
<tr>
<td>n/a</td>
<td>0.0450</td>
<td>0.0450</td>
<td>Two Loops</td>
</tr>
<tr>
<td>0.052</td>
<td>n/a</td>
<td>0.052</td>
<td>Two Loops</td>
</tr>
<tr>
<td>0.043</td>
<td>0.043</td>
<td>0.043</td>
<td>Simultaneous</td>
</tr>
</tbody>
</table>

that end, consider the following low-pass filter

$$C_1(s) = \frac{1}{0.05s+1} \frac{(-6s+1)^2}{(8s+1)^2},$$

for which the $L_1$ gain requirement holds. The corresponding $D_1(s)$ will be

$$D_1(s) = \frac{(-6s+1)^2}{s(3.2s^2 + 28.8s + 28.05)}.$$

The Bode plots of $D(s)$ and $D_1(s)$ are given in Figure 6. Note that a nonminimum phase filter is used to enhance the phase characteristic in the region of frequency-band in order to improve the phase margin.

Figure 6. Choice of $C(s)$ to maximize the time-delay margin
The subplot of vertical acceleration of Figure 1 is repeated with $C_1(s)$ in Figure 7. We see that there is some degradation in the tracking, however it is still satisfactory. For this choice of $C_1(s)$, the worst case time-delay obtained from simulation is 0.10 sec. In Table 2, we have the worst case time-delay margin for $C(s)$ equal to 0.0425 sec, which implies that $C_1(s)$ doubles the time-delay margin. Thus, at this stage, it appears that improving the time-delay margin hurts the transient performance, which is consistent with the conventional claims in linear systems theory. From this perspective, the $\mathcal{L}_1$ adaptive control paradigm achieves clear separation between adaptation and robustness: the adaptation can be as fast as the CPU permits, while robustness can be resolved via conventional methods well known from classical and robust control. We note that all the time-delay margins are calculated for the ROB actuator failure case and in the presence of the pitch break uncertainty.

![Figure 7. Inner-Loop Adaptation with ROB Elevon Failure and Pitch Break Phenomenon: Command Tracking (for Different Choice of $C(s)$)](image)

However, we note that a smaller value of $\Gamma$ is preferable from an implementation point of view. Figure 8 shows the system response for different values of $\Gamma$ for both low-pass filters. It can be seen that there is almost no degradation in the time response performance. However, for the low-pass filter $C(s) = \frac{1}{0.05s+1}$, if we decrease $\Gamma$ from 100000 to 10000, the worst case time-delay margin decreases from 0.043 to 0.003 (i.e. almost 14 times). However, for $C_1(s)$ it decreases from 0.11 to 0.09 (i.e. only 1.2 times). Thus, with smaller choice of $\Gamma$, $C_1(s)$ is much suitable in terms of robustness as compared to $C(s)$. Table 3 summarizes the margins for $C_1(s)$ with $\Gamma = 10000$.

Finally, we simulate the system with two actuator failures without any delays to test robustness of the $\mathcal{L}_1$ adaptive control scheme towards a different class of uncertainty. Figure 9 plots the vertical acceleration command tracking in the presence of pitch break uncertainty and ROB and LMB elevon failures. We see that the $\mathcal{L}_1$ adaptive control architecture retains both its tracking property and as well the worst time-delay margin of 0.095, as predicted by the
Table 3. Time-delay margin of $\mathcal{L}_1$ for $C_1(s)$ and $\Gamma = 10000$

| $\dot{p}$ | $\dot{q}$ | $\dot{r}$ |\n|---|---|---|\n| n/a | 0.18 | n/a | Individual |\n| 0.3 | n/a | n/a | Individual |\n| n/a | n/a | 0.25 | Individual |\n| 0.10 | 0.10 | n/a | Two Loops |\n| n/a | 0.10 | 0.10 | Two Loops |\n| 0.12 | n/a | 0.12 | Two Loops |\n| 0.09 | 0.09 | 0.09 | Simultaneous |

Figure 8. Inner-Loop Adaptation with ROB Elevon Failure and Pitch Break Phenomenon: Command Tracking (for different values of $\Gamma$)

$L_1$ theory. In Table 4, we have summarized the worst-case time delay margins for single actuator and two actuator failures.
Figure 9. MRAC loses stability in the presence of two actuator failures, while $\mathcal{L}_1$ proves guaranteed tracking

Table 4. Worst case time-delay margin with single and double actuator failures

<table>
<thead>
<tr>
<th></th>
<th>Single</th>
<th>Double</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma$</td>
<td>0.043</td>
<td>0.043</td>
<td>C(s) 100000</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>0.0225</td>
<td>0.020</td>
<td>C(s) 10000</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>0.11</td>
<td>0.13</td>
<td>$C_1(s)$ 100000</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>0.09</td>
<td>0.095</td>
<td>$C_1(s)$ 10000</td>
</tr>
</tbody>
</table>

VI. Aerial Refueling

In this section we apply the $\mathcal{L}_1$ adaptive controller to Autonomous Aerial Refueling (AAR) autopilot design. The probe-and-drogue refueling procedure is adopted. This has proven to be extremely difficult due to the aerodynamic coupling among the two aircraft, receiver and tanker, and the drogue. The autopilot has to compensate for the uncertainties due to the trailing vortices of the tanker when the receiver is flying from the observation point to the contact point. As compared to the earlier work of the authors [19], the control signal defined via (27), (28) and (30) subject to the $\mathcal{L}_1$-gain stability requirement in (26), eliminates the need for selecting and tuning basis functions required by neural network based $\mathcal{L}_1$ adaptive control. This reduces the design effort significantly.

We consider a decoupled linearized six-degree-of-freedom (6-DOF) aircraft model for the receiver aircraft. Both the receiver and the tanker aircraft are in straight and level flight in the beginning of the maneuver. The receiver is
assumed to be subject only to small perturbations during the entire refueling maneuver so that the linearized decoupled dynamics can be used to describe its motion with some level of fidelity, with additive uncertainties coming from the trailing vortices induced by the tanker aircraft. For the aerial refueling maneuver all angles are assumed small. Besides these assumptions, we also neglect the influence of gravity, thrust, and elevator on the angle of attack. The drogue position is known, served as reference command \( z_d(t) \), and the control objective is to fly the receiver into a prescribed neighborhood of the drogue center within a prescribed finite time interval. Upon incorporating inner loop baseline controllers into the system dynamics, the system to be controlled has three inputs: thrust, elevator and aileron, and has three outputs: horizontal separation, vertical separation and lateral separation relative to the tanker. The control objective is to regulate three outputs to certain set points in the presence of aerodynamic uncertainties (drag, pitching moment and rolling moments) in three directions respectively, and in the presence of control surface failures.

The aircraft model is a flying-wing unmanned aerial vehicle (UAV), known as the Barrons Associates nonlinear tailless aircraft model (BANTAM) [14]. The wake-effect data which models the vortex are taken from a wind-tunnel test of a delta wing UAV behind KC-135R tanker [20]. The baseline controller is a Linear Quadratic Regulation (LQR) controller with integral action. For this LQR+PI controller structure, the feedforward gain \( K_z \) is zero. The system takes the form in (16), which is repeated here for the sake of completeness:

\[
\dot{x}(t) = A_m x(t) + B_m z_c(t) + B_1 \left( \Lambda u_{ad}(t) + \Lambda K_0(t, x_p(t)) + k_x^T x(t) \right),
\]

(61)

where \( x(t) \in \mathbb{R}^{14}, u_{ad}(t) \in \mathbb{R}^3 \) (thrust, elevator and aileron), \( z_c(t) \in \mathbb{R}^3 \) are the measured system states, control signals and reference inputs, respectively, \( A_m \in \mathbb{R}^{14 \times 14}, B_m \in \mathbb{R}^{14 \times 3}, B_1 \in \mathbb{R}^{14 \times 3} \) are known matrices where the three columns of \( B_1 \) are linearly independent. The definitions of \( \Lambda, k_x \) are given in (17). Notice that there is no \( k_z \) in equation (61). The state vector \( x = (l, V, \alpha, \theta, q, h, \phi, \beta, p, r, y, l_f, h_f, y_f)^T \) comprises eleven plant states \( (x_p) \), which include horizontal separation \( l \), velocity \( V \), angle of attack \( \alpha \), pitch angle \( \theta \), pitch rate \( q \), vertical separation \( h \), roll angle \( \phi \), angle of sideslip \( \beta \), body roll rate \( p \), body yaw rate \( r \), lateral separation \( y \) and three baseline controller \( (x_c) \) states, which include integrator states of separations \( (l_f, h_f \) and \( y_f) \). The simulation results and \( L_1 \) controller parameters are given in this section. More details of system dynamics and baseline controllers can be found in [19].

The target point for the receiver aircraft is chosen to be the center of the outer cross section of the drogue. The aircraft is trimmed at speed \( V_0 = 500 \text{ ft/sec} \), angle of attack \( \alpha_0 = 0.042 \text{ rad} \), pitch angle \( \theta_0 = 0.042 \text{ rad} \), and at the altitude of \( h = 5000 \text{ ft} \). The radius of the drogue is \( r_d = 1 \text{ ft} \). The initial position of the receiver aircraft is 165 ft behind the tanker and 50 ft below the tanker, and 10 ft to the left of the aircraft laterally. Relative to the tanker coordinate system the position of the drogue center is at the coordinates: \( x_d = -15 \text{ ft}, y_d = 30 \text{ ft}, z_d = 10 \text{ ft} \).

The closed loop system with these baseline controller gains defines the nominal linear system response that has the desired convergence time for the probe to contact the drogue with desired performance specifications. The adaptive augmentation with the \( L_1 \) controller is designed to track this system’s response both in transient and steady state. In the absence of wake induced uncertainties and without any loss in control effectiveness, the probe reaches the 0.02 ft neighborhood of the drogue center within 25 sec. That is, \(|x(25) - x_d(25)| = 0.012, |y(25) - y_d(25)| = 0.015\).
and $|z(25) - z_d(25)| = 0.02$. Figure 10 plots the closed-loop trajectories in each axis. It compares the responses in the absence of the wake and without any loss in control effectiveness to the response in the presence of the wake and loss in control effectiveness. The failures are 60% reduction of elevator effectiveness, and 60% reduction of aileron effectiveness, that is $\Lambda_2 = \Lambda_3 = 0.4$. From these figures, we can see that both the steady-state tracking and the transient performance are deteriorated in the presence of uncertainties. Figure 11 shows the other states (angle of attack $\alpha$, pitch angle $\theta$, roll angle $\phi$, side-slip angle $\beta$, roll rate $p$, pitch rate $q$ and yaw rate $r$) of the baseline controlled closed loop system. Those states remain small during the whole aerial refueling process.

For the design of $L_1$ controller we choose $D_i(s) = \frac{1}{s}$, $i = 1, \ldots, m$. Conservative growth rates for the uncertainties are computed from the experimental data, implying that $\max L_{w_i} = 0.1$. The conservative maximum is given by $L_i = 10.1$. We choose $k_1 = k_2 = k_3 = 20$ leading to $C_1(s) = \frac{20}{s+20}$, $C_2(s) = \frac{8}{s+8}$ and $C_3(s) = \frac{8}{s+8}$. We set a uniform adaptive gain $\Gamma = 100000$ for the design of the adaptive controller in each axis. The closed loop system response with $L_1$ adaptive augmentation and the adaptive control signals are shown in Figure. 12. We notice that $L_1$ adaptive controller can recover the nominal performance of the baseline controller in the presence of the wake vortex and loss in control effectiveness. The tracking precision upon $t' = 25\text{sec}$ can be characterized with the bound $\| [y(t') \ z(t')]^T - [y_d(t') \ z_d(t')]^T \| = 0.05 \leq 1$, and $|x(t') - x_d(t')| = 0.04$. Notice that the in the thrust input channel, at the steady state there is a difference between the time histories of the adaptive approximation and the uncertainty. That is caused by the residue of the elevator control input coming into the horizontal direction. In the elevator and aileron plots, we see that there are some differences between the adaptive signals and the uncertainties.
during the transient phase. If we turn off the actuator failure, the adaptive approximation will be much closer to the uncertainties, as shown in Figure 13.

Next we show that when the initial conditions change, $L_1$ controller does not need any re-tuning.

We change the initial conditions of the receiver. We keep the same values for $\Lambda_1$ and $\Lambda_2$, and run the simulations from 33.75 ft behind the tanker, 18 ft below the tanker and 22 ft to the right of the tanker, without any re-tuning of both controllers. The $L_1$ adaptive controller achieves the tracking precision upon $t' = 25$ sec, which is $\| [y(t') \quad z(t')] \|^2 = 0.021 \leq 1$, and $|x(t') - x_d(t')| = 0.020$, as in Fig. 14. We observe that $L_1$ adaptive controller shows scaled system output responses (similar to linear systems).

Without re-tuning of the $L_1$ controller, we further run the simulation starting from various different initial con-
ditions, as shown in Fig 15. It can be seen that the system outputs have scaled responses. We further increase the

uncertainties, by multiplying the magnitude of the vortex data by two. This can approximately represent a different tanker if we consider a modified horseshoe vortex model. With fixed separations between the receiver and the tanker, the magnitude of the induced velocity is proportional to the strength of the vortex, which is in turn proportional to the mean aerodynamic chord of the tanker and velocity of the tanker. So it is reasonable to scale our current induced aerodynamic coefficients by multiplying them by two to represent the change of tanker’s wake effects. Figure 16 shows that the the outputs are identical to the nominal performance, and Fig. 17 shows that the adaptive control signals have scaled responses corresponding to the change of uncertainties.

Finally, we compute the time-delay margin of the $L_1$ adaptive controlled system similar to the UCAV model above.
First we use the low pass filters $C_2(s) = C_3(s) = \frac{25}{s + 25}$. The numerical values are shown in Table 5.

<table>
<thead>
<tr>
<th>Elevator</th>
<th>Aileron</th>
<th>Delay Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.028</td>
<td>n/a</td>
<td>Individual</td>
</tr>
<tr>
<td>n/a</td>
<td>0.031</td>
<td>Individual</td>
</tr>
<tr>
<td>0.028</td>
<td>0.030</td>
<td>Simultaneous</td>
</tr>
</tbody>
</table>

We next change the low pass filter to $C_2(s) = C_3(s) = \frac{8}{s + 8}$. The time-delay margins are given in Table 6.

<table>
<thead>
<tr>
<th>Elevator</th>
<th>Aileron</th>
<th>Delay Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.042</td>
<td>n/a</td>
<td>Individual</td>
</tr>
<tr>
<td>n/a</td>
<td>0.081</td>
<td>Individual</td>
</tr>
<tr>
<td>0.041</td>
<td>0.080</td>
<td>Simultaneous</td>
</tr>
</tbody>
</table>

We see that by applying different $C(s)$ we can improve the time delay margin of the adaptive controlled system. This verifies the theoretical predictions.

## VII. Conclusion

The $L_1$ adaptive controller is applied to two benchmark flight control applications in this paper. The proposed adaptive control approach overcomes the drawbacks of conventional adaptive control methods. It has guaranteed performance bounds and systematic design methodology to achieve the desired control specifications. The bounded-away-from-zero time delay margin of this adaptive controller can be improved by systematic choice of the underlying filters. These features hold a promise for the development of theoretically justified tools for Validation and Verification of adaptive systems.

## VIII. Acknowledgment

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We recall some definitions and facts from linear systems theory [21–23].

**Definition 1** For a signal \( \xi(t) \), \( \xi \in \mathbb{R}^n \), its truncated \( L_\infty \) norm and \( L_\infty \) norm are defined as

\[
\| \xi \|_{L_\infty} = \max_{i=1, \ldots, n} \left( \sup_{0 \leq \tau \leq t} |\xi_i(\tau)| \right),
\]

\[
\| \xi \|_{L_\infty} = \max_{i=1, \ldots, n} \left( \sup_{\tau \geq 0} |\xi_i(\tau)| \right),
\]

where \( \xi_i \) is the \( i \)th component of \( \xi \).

**Definition 2** The \( L_1 \) gain of a stable proper single–input single–output system \( H(s) \) is defined to be \( \|H(s)\|_{L_1} = \int_0^\infty |h(t)| dt \), where \( h(t) \) is the impulse response of \( H(s) \).

**Definition 3** For a stable proper \( m \) input \( n \) output system \( H(s) \) its \( L_1 \) gain is defined as

\[
\|H(s)\|_{L_1} = \max_{i=1, \ldots, n} \left( \sum_{j=1}^m \|H_{ij}(s)\|_{L_1} \right),
\]

where \( H_{ij}(s) \) is the \( i \)th row \( j \)th column element of \( H(s) \).

**References**


